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BRIEF COMMUNICATION

On the generalization of the Edgeworth/Gram-Charlier series

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Abstract Berberan-Santos relates one probability distribution to another wherein one distribution is expressed in terms of the derivatives of the other Berberan-Santos (J Math Chem 42:585, 2007). We derive the result in a straightforward manner based on a generalization of the Edgeworth/Gram Charlier.

Keywords Edgeworth series · Gram-Charlier series · Probability density

In [1] Berberan-Santos obtained an interesting result where a probability distribution is expanded in terms of the derivatives of another distribution and where the coefficients of the expansion involves the cumulants. The aim of this paper is to show that his result can be obtained in a simple and direct manner using the generalization of the Edgeworth/Gram-Charlier series that the author has previously given [2–4]. We first describe a general approach relating probability distributions and then specialize to the case considered by Berberan-Santos. Any two probability distributions $P_1(x)$ and $P_2(x)$, may be related by [2–4]

$$P_2(x) = \Omega(\mathcal{A}) P_1(x) \tag{1}$$

where $\mathcal{A}$ is any hermitian operator [5] and Ω is a function of the operator $\mathcal{A}$. Of particular interest here is where

$$\mathcal{A} = iD \tag{2}$$

and where D is the differentiation operator

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$$D = \frac{d}{dx} \tag{3}$$

For this case

$$P_2(x) = \ \Omega(iD)P_1(x) \tag{4}$$

and

$$\Omega(\theta) = \frac{M_2(\theta)}{M_1(\theta)} \tag{5}$$

where $M_1(\theta)$ and $M_1(\theta)$ are the characteristic functions of the respective distributions,

$$M_1(\theta) \ = \int e^{i\theta x} P_1(x)\,dx \tag{6}$$

$$M_2(\theta) \ = \int e^{i\theta x} P_2(x)\,dx \tag{7}$$

The proof of Eq. (4)–(5) is given in references [2–4] where it was shown that expressing the characteristic functions in different forms and taking P_1 to be the normal distribution results in the classical Edgeworth series and Gram-Charlier series [2,6]. To obtain the result of Berberan-Santos we express the characteristic function in terms of its cumulants, κ_n

$$M(\theta) = \exp\left[\sum_{n=1}^{\infty} \kappa_n \frac{i^n}{n!}\theta^n\right] \tag{8}$$

Therefore, using Eq. (4)–(5), we have

$$\Omega(iD) = \frac{M_2(iD)}{M_1(iD)} = \exp\left[\sum_{n=1}^{\infty} \left(\kappa_n^{(2)} - \kappa_n^{(1)}\right) \frac{(-1)^n}{n!} D^n\right] \tag{9}$$

and

$$P_2(x) = \exp\left[\sum_{n=1}^{\infty} \left(\kappa_n^{(2)} - \kappa_n^{(1)}\right) \frac{(-1)^n}{n!} D^n\right] P_1(x) \tag{10}$$

In Eq. (9) $\kappa_n^{(1)}$ and $\kappa_n^{(2)}$ are the respective cumulants of P_1 and P_2. This is a general expression relating any two distributions and its cumulants which has previously been derived [2].

To obtain the explicit result of [1] we consider expressing

$$f(z) = \exp\left[\sum_{n=1}^{\infty} \delta_n \frac{(-1)^n}{n!} z^n\right] \tag{11}$$

in a power series,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \tag{12}$$

where $f^{(n)}$ is the nth derivative of the function and where we define δ_n as in [1]

$$\delta_n = \kappa_n^{(2)} - \kappa_n^{(1)} \tag{13}$$

To obtain $f^{(n)}(0)$ we start with the somewhat more general form

$$f(z) = e^{g(z)} \tag{14}$$

Differentiating Eq. (14) we have

$$f^{(1)} = g^{(1)} f \tag{15}$$

and using Leibniz's rule we have

$$f^{(n+1)}(z) = \sum_{k=0}^{n} \binom{n}{k} g^{(n+1-k)}(z) f^{(k)}(z) \tag{16}$$

In our case

$$g(z) = \sum_{n=1}^{\infty} \delta_n \frac{(-1)^n}{n!} z^n \tag{17}$$

and hence

$$g^{(n)}(0) = (-1)^n \delta_n \tag{18}$$

Substituting this into Eq. (16) we have that

$$f^{(n+1)}(0) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{(n+1-k)} \delta_{n+1-k} f^{(k)}(0) \tag{19}$$

This is a recurrence relation that generates $f^{(n)}(0)$. Letting $z = D$, Eq. (10) becomes

$$P_2(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} D^n P_1(x) \tag{20}$$

This is the result of [1] except for notation. To use the notation of [1] one defines

$$\alpha_n = (-1)^n f^{(n)}(0) \tag{21}$$

Then

$$\alpha_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \delta_{n+1-k}\alpha_k = \delta_{n+1} + \delta_1\alpha_n + \sum_{k=1}^{n-1} \binom{n}{k} \delta_{n+1-k}\alpha_k \tag{22}$$

and

$$P_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \alpha_n}{n!} P_1^{(n)} \tag{23}$$

The explicit values for the α_n's as given in [1] are readily generated using Eq. (22) with $\alpha_0 = 1$. In particular, the first few α_n's are $\alpha_1 = \delta_1, \alpha_2 = \delta_1^2 + \delta_2$, and $\alpha_3 = \delta_1^3 + 3\delta_1\delta_2 + \delta_3$.We also point out that Eq. (23) can be generalized in that for any Hermitian operator, $\mathcal{A}$, one has

$$P_2(x) = \sum_{n=0}^{\infty} \beta_n \mathcal{A}^n P_1(x) \tag{24}$$

Methods to obtain the coefficients β_n will be developed in another paper.

References

1. M.N. Berberan-Santos, J. Math. Chem. **42**, 585 (2007)
2. L. Cohen, Multidimensional Syst. Signal Process. **9**, 363–372 (1998)
3. L. Cohen, Integr. Comput. Aided Eng. **12**(1), 129–134 (2005)
4. L. Cohen, Proc. SPIE **6566**, 15–18 (2007)
5. L. Cohen, *Time-Frequency Analysis* (Prentice-Hall, Englewood Cliffs, 1995)
6. A. Stuart, K.K. Ord, *Kendall's Advanced Theory of Statistics* (Halsted Press, 1994)